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RESEARCH ARTICLE

Analogue of the Cole-Hopf transform for the incompressible Navier-Stokes equations and its application

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We consider the Navier-Stokes equations written in the stream function in two dimensions and vector potentials in three dimensions, which are critical dependent variables. On this basis, we introduce an analogue of the Cole-Hopf transform, which exactly reduces the Navier-Stokes equations to the heat equations with a potential term (i.e. the nonlinear Schrödinger equation at imaginary times). The following results are obtained. i) A regularity criterion immediately obtains as the boundedness of condition for the potential term when the equations are recast in a path-integral form by the Feynman-Kac formula. ii) This in turn gives an additional characterisation of possible singularities for the Navier-Stokes equations. iii) Some numerical results for the two-dimensional Navier-Stokes equations are presented to demonstrate how the potential term captures near-singular structures. Finally, we extend this formulation to higher dimensions, where the regularity issues are markedly open.

Keywords:

Navier-Stokes equations, Cole-Hopf transform, Feynman-Kac formula, Duhamel principle

1. Introduction

It is well-known that the Cole-Hopf transform [1, 2] linearises the Burgers equation into the heat equation, thereby allowing an exact solution to its initial-value problem. This is the case in any spatial dimensions, provided that we restrict ourselves to potential flows. The derivation by Cole [1] explicitly makes use of the idea of critical scale-invariance of the dependent variable, the velocity potential.

On the other hand, while global regularity is established for the two-dimensional Navier-Stokes equations, the proof is not given by linearisation, but it relies e.g. on the fact that the total kinetic energy is critical in two dimensions.

In this paper we first seek an analogue of the Cole-Hopf transform for the Navier-Stokes equations and its applications in two-dimensional turbulence. We neither expect linearisation nor complete integrability for the Navier-Stokes equations. Rather, we will show by a simple analysis that it is possible to rewrite them in the form of the nonlinear Schrödinger equation by introducing an analogue of the Cole-Hopf transform. It does not give a proof of regularity, but does yield a criterion for regularity through a conversion to an integral equation by the Feynman-Kac formula. It also offers an alternative method for detecting near-singularities in turbulent flows.

In Section 2, as an illustration of the main idea, we recall how the Burgers equations with a forcing term can be linearised and solved by a path integral formulation. In Section 3, by showing how a similar formulation is applicable, we

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introduce an analogue of the Cole-Hopf transform for the two-dimensional Navier-Stokes equations. In Section 4, some numerical results are presented to show how such an approach is useful in the numerical study of two-dimensional turbulence. In Section 5, by using the formulation based on the vector potential, we generalise the formulation to the three-dimensional Navier-Stokes equations. Section 6 is devoted to summary and discussion, where a new characterisation of possible singularities in three dimensions is given. Appendices include some technical details of the potential term f, those of the nonlinear term $T[\nabla\psi]$ in two dimensions and an extension to higher dimensions.

2. Burgers equation

This section describes a motivation for the current approach and it does not include new materials.

We consider the Burgers equations in $\mathbb{R}^n \quad (n \geq 1)$ with an external forcing of the form $-\nabla V(\boldsymbol{x}, t)$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla V + \nu\triangle\boldsymbol{v}, \qquad (1)$$
$$\boldsymbol{v}(\boldsymbol{x},0) = \boldsymbol{v}_0(\boldsymbol{x}).$$

Here we restrict ourselves to the special class of irrotational flows $\boldsymbol{v} = \nabla\phi$. The variable $\boldsymbol{v}$ satisfies the following well-known scale-invariance: if $\boldsymbol{v}(\boldsymbol{x}, t)$ is a solution, then so is $\lambda\boldsymbol{v}(\lambda\boldsymbol{x}, \lambda^2 t)$ for any $\lambda > 0$. The equation (1) can be integrated to give

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + V = \nu\triangle\phi, \qquad (2)$$
$$\phi(\boldsymbol{x},0) = \phi_0(\boldsymbol{x}).$$

where the constant of integration is taken to be zero. The variable ϕ satisfies the following scale-invariance: if $\phi(\boldsymbol{x}, t)$ is a solution, then so is $\phi(\lambda\boldsymbol{x}, \lambda^2 t)$ for any $\lambda > 0$. It is critical in the sense that ϕ lacks a prefactor after the transformation. This stems from the fact that ϕ has the same physical dimension as kinematic viscosity ν. Applying a transform $\phi = k\log\theta$, we rewrite (2) as

$$\frac{\partial \theta}{\partial t} = \nu\triangle\theta - \left(\frac{k+2\nu}{2}\frac{|\nabla\theta|^2}{\theta^2} + \frac{V}{k}\right)\theta,$$
$$\theta(\boldsymbol{x},0) = \theta_0(\boldsymbol{x}).$$

By the standard choice[1, 2] $k = -2\nu$, we can linearise the above to obtain a heat equation with a potential term, i.e. the nonlinear Schrödinger equation at imaginary times

$$\frac{\partial \theta}{\partial t} = \nu\triangle\theta + \frac{1}{2\nu}V\theta, \qquad (3)$$

see e.g [3, 4]. If $V \equiv 0$, it reduces to the simplest form of the heat equation, which can be solved as

$$\theta(\boldsymbol{x},t) = \frac{1}{(4\pi\nu t)^{n/2}}\int_{\mathbb{R}^n}\exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4\nu t}\right)\theta_0(\boldsymbol{y})d\boldsymbol{y}$$
$$= G_t * \theta_0, \qquad (4)$$

where $G_t = \frac{1}{(4\pi\nu t)^{n/2}}\exp\left(-\frac{|\boldsymbol{x}|^2}{4\nu t}\right)$ denotes the heat kernel and $*$ a convolution. In more general cases where $V \neq 0$, under a condition of boundedness $\int_0^t \sup_{\boldsymbol{x}} |V(\boldsymbol{x},s)|ds < \infty$, (3) is solvable using the Feynman-Kac formula

$$\theta(\boldsymbol{x},t) = E\left[\theta_0(\boldsymbol{B}_t)\exp\left(\frac{1}{2\nu}\int_0^t V(\boldsymbol{B}_s,s)ds\right)\right].$$

Here $\boldsymbol{B}_t = \sqrt{2\nu}\boldsymbol{W}_t$ with $\boldsymbol{W}_t$ denoting an n-dimensional standard Brownian motion starting from $\boldsymbol{x}$ at $t=0$ and $E[\cdot]$ an average with respect to a probability measure associated with $\boldsymbol{B}_t$.

3. Cole-Hopf transform for the two-dimensional Navier-Stokes equations

In this and next sections, we consider the two-dimensional incompressible Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + \nu\triangle\boldsymbol{u}, \qquad (5)$$
$$\nabla\cdot\boldsymbol{u} = 0,$$
$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}),$$

where the velocity $\boldsymbol{u} = (\partial_2\psi, -\partial_1\psi)$ has the stream function ψ. In term of vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$, the governing equation can also be written

$$\frac{\partial\omega}{\partial t} + \boldsymbol{u}\cdot\nabla\omega = \nu\triangle\omega, \qquad (6)$$

or, by $\omega = -\triangle\psi$, it reads

$$\frac{\partial\triangle\psi}{\partial t} + \frac{\partial(\triangle\psi,\psi)}{\partial(x_1,x_2)} = \nu\triangle^2\psi.$$

By applying the inverse Laplacian operator $\triangle^{-1}$, the governing equation for ψ has been derived in [5]

$$\frac{\partial\psi}{\partial t} = T[\nabla\psi] + \nu\triangle\psi. \qquad (7)$$

Here we have denoted the nonlinear term by

$$T[\nabla\psi] = \frac{1}{\pi}\text{P.V.}\int_{\mathbb{R}^2}\frac{[(\boldsymbol{x}-\boldsymbol{y})\times\nabla\psi(\boldsymbol{y})]\,(\boldsymbol{x}-\boldsymbol{y})\cdot\nabla\psi(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^4}\,\mathrm{d}\boldsymbol{y}, \qquad (8)$$

where P.V. stands for a principal-value integral. An analogue of the Cole-Hopf transform for this equation can be obtained by a simple analysis as follows.

Proposition 3.1: *We can rewrite the two-dimensional Navier-Stokes equations in a form similar to (3):*

$$\frac{\partial \theta}{\partial t} = \nu \triangle \theta + f(\boldsymbol{x}, t)\theta, \tag{9}$$

where

$$f(\boldsymbol{x}, t) = k\, T\left[\frac{\nabla \theta}{\theta}\right] - \nu \frac{|\nabla \theta|^2}{\theta^2}. \tag{10}$$

Proof: It is done by a straightforward calculation. We set $\psi = k \log \theta$ for a strictly positive function $\theta(\boldsymbol{x}, t) > 0$ and a real parameter $k \neq 0$, whose physical dimension is the same as ν, and compute in a spirit similar to [3]

$$\begin{aligned}
\psi_t - T[\nabla \psi] - \nu \triangle \psi &= k\frac{\theta_t}{\theta} - k^2 T\left[\frac{\nabla \theta}{\theta}\right] - \nu k \left(\frac{\triangle \theta}{\theta} - \frac{|\nabla \theta|^2}{\theta^2}\right) \\
&= k \left\{ \frac{\theta_t - \nu \triangle \theta}{\theta} - \underbrace{\left(k\, T\left[\frac{\nabla \theta}{\theta}\right] - \nu \frac{|\nabla \theta|^2}{\theta^2}\right)}_{\equiv f(\boldsymbol{x}, t)} \right\}.
\end{aligned}$$

By setting the right-hand side to be zero, we find (9) together with (10). □

The equation (9) can be regarded as an exact reduction[1] to the nonlinear Schrödinger equation at imaginary times.

We observe that the function V in the Burgers equation (1) corresponds, at least formally, to the pressure p in the Navier-Stokes equations (5). Hence (9) suggests that f corresponds to p. In fact, they are close, but not exactly identical as we will see in Appendix **A**. Just like (3), the equation (9) is the heat equation subject to a potential term f, but which itself depends on the unknown ψ. Note that in the case of the Burgers equations without forcing, the potential term vanishes because of complete cancellations and linearization is achieved. Here, f remains non-zero because of mismatch of nonlinear terms and this in turn gives rise to a condition for the regularity of their solutions. To see this more precisely, we apply a kind of Duhamel principle to rewrite (9) as an integral equation with use of the Feynman-Kac formula.

Proposition 3.2: *By regarding f as a given external forcing, we recast the above equation (9) as a path integral using the Feynman-Kac formula*

$$\theta(\boldsymbol{x}, t) = E\left[\theta_0(\boldsymbol{B}_t) \exp\left(\int_0^t f(\boldsymbol{B}_s, s) ds\right)\right]. \tag{11}$$

Proof: It suffices to check how the usual proof (e.g. [7], also [8]) works even when the potential f depends on the unknown variable. Consider an interpolating martingale

$$M_s \equiv \theta(\boldsymbol{B}_s, t - s) \exp\left(\int_0^s f(\boldsymbol{B}_r, r) dr\right), \quad 0 \leq s \leq t.$$

[1]For an inviscid fluid, the well-known Madelung transform connects the nonlinear Schrödinger equation (at real times) with the Euler equations with the quantum pressure term. See [6] and references cited therein.

We have by Itô formula

$$\begin{aligned} d\theta(\boldsymbol{B}_s, t-s) &= d\boldsymbol{B}_s \cdot \nabla\theta(\boldsymbol{B}_s, t-s) + \nu\triangle\theta(\boldsymbol{B}_s, t-s)ds - \partial_t\theta(\boldsymbol{B}_s, t-s)ds \\ &= d\boldsymbol{B}_s \cdot \nabla\theta(\boldsymbol{B}_s, t-s) - f(\boldsymbol{B}_s, s)\theta(\boldsymbol{B}_s, t-s)ds. \end{aligned}$$

Then we find

$$\begin{aligned} dM_s &= \exp\left(\int_0^s f(\boldsymbol{B}_r, r)dr\right)(d\boldsymbol{B}_s \cdot \nabla\theta(\boldsymbol{B}_s, t-s) - f(\boldsymbol{B}_s, s)\theta(\boldsymbol{B}_s, t-s)ds) \\ &+ \exp\left(\int_0^s f(\boldsymbol{B}_r, r)dr\right)\theta(\boldsymbol{B}_s, t-s)f(\boldsymbol{B}_s, s)ds \\ &= \exp\left(\int_0^s f(\boldsymbol{B}_r, r)dr\right)d\boldsymbol{B}_s \cdot \nabla\theta(\boldsymbol{B}_s, t-s), \end{aligned} \tag{12}$$

which shows that M_s is a local martingale.

Moreover, if θ and f are bounded from above

$$\sup_{0\le s\le t}|M_s| \le \|\theta\|_{L^\infty}\exp\left(\int_0^t \|f\|_{L^\infty}ds\right),$$

M_s is a martingale. We then have $E[M_0] = E[M_t]$, from which (11) follows. □

While global regularity for the two-dimensional Navier-Stokes equation is well-established, it is of interest to identify from (11) that the boundedness of $\int_0^t \sup_{\boldsymbol{x}}|f(\boldsymbol{x}, s)|ds$ is necessary and sufficient for the regularity of solutions of the Navier-Stokes equations on $[0, t]$. That means, if f neither blows up to ∞, nor blows down to $-\infty$ in finite time, the solutions remain smooth. In particular, the latter condition is important because it is required for θ not to hit 0. (Recall $\psi = k\log\theta$.) In Section 6 we compare the above formula with the existing regularity criteria, after working out the three-dimensional case in section 5.

4. Numerical experiments in two-dimensions

Now that we have converted the two-dimensional Navier-Stokes equation in a form of a path integral equation (11), it is of interest to check how the potential term f behaves in numerical simulations of turbulent flows. We recall that although global regularity of their solutions is guaranteed mathematically, the vorticity gradient can and in practice do become large in turbulent flows, thereby generating near-singularities. Hence it is of interest to observe this phenomenon from the probabilistic approach we have just introduced.

We have carried out direct numerical simulations of the Navier-Stokes equations using a standard Fourier pseudo-spectral method under periodic boundary conditions. The grid points N^2 used are: $N = 1024$ for $\nu = 2\times10^{-3}$, $N = 2048$ for $\nu = 1\times10^{-3}$ and $N = 4096$ for $\nu = 5\times10^{-4}$. Aliasing errors were removed by the so-called 2/3-law. The time marching was done by a fourth-order Runge-Kutta scheme with $\Delta t = 2\times10^{-3}$. In practice we solve (6) numerically by the standard method and from ω thus obtained we compute $\psi = -\triangle^{-1}\omega$ and $T[\nabla\psi]$ by post-processing in wavenumber space. See **Appendix** B for more details.

The initial condition was chosen as

$$\omega(\boldsymbol{x}, 0) = \cos(2x)\cos(y) + \sin(x)\sin(y) + \cos(2x)\sin(3y),$$

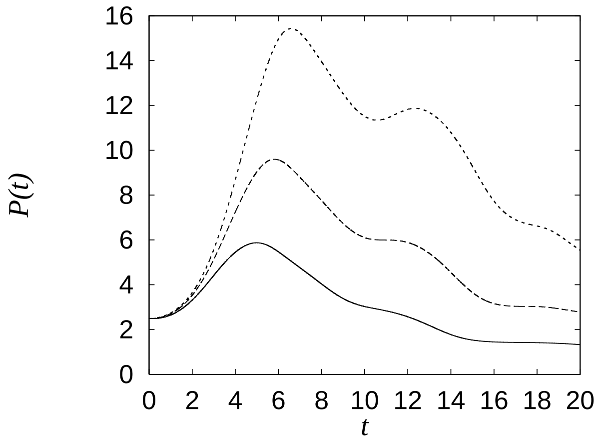

Figure 1. Time evolution of palinstrophy $P(t)$ for $\nu = 2 \times 10^{-3}$ (solid), $\nu = 1 \times 10^{-3}$ (dashed), and $\nu = 5 \times 10^{-4}$ (dotted).

which is one of those used in [9].

In this freely-decaying computation, both the total kinetic energy $E(t) = \frac{1}{2}\left\langle |\boldsymbol{u}|^2 \right\rangle$ and the enstrophy $Q(t) = \frac{1}{2}\left\langle \omega^2 \right\rangle$ decrease monotonically in time, where the brackets denote a spatial average on the domain $[0, 2\pi]^2$. The palinstrophy $P(t)$, i.e. a spatial average of squared vorticity gradient,

$$P(t) = \frac{1}{2}\left\langle |\nabla\omega|^2 \right\rangle$$

attains a peak value around $t = 5$ and decays thereafter, see Fig.1. Let us define the Reynolds number by $R = \frac{UL}{\nu}$, where $U = \sqrt{\frac{\langle |\boldsymbol{u}|^2 \rangle}{2}} = \sqrt{E}$ denotes the characteristic velocity and $L = 2\pi$ the characteristic length scale. For example, with $\nu = 1 \times 10^{-3}$ we have $R = 1958$ initially and $R = 1796$ at the end of $t = 40$. Throughout the computation, the Reynolds number remains at a moderately high level and characteristic features observed in turbulence with higher R are also observed here.

In Fig.2, we present gradation plots of the vorticity ω, the stream function ψ, the nonlinear term $T[\nabla\psi]$ and its excess over local kinetic energy $T[\nabla\psi] - |\nabla\psi|^2$, at $t = 8$ for the case of $N = 1024$. (Note that the latter corresponds to a choice of $k = 1/2$, but the qualitative features of f do not depend on k strongly as long as $k > 0$.) Two positive and two negative vortices are seen in ω. We observe a smoother pattern in ψ, which is expected from its lower-order differentiation. Their locations of characteristic structures with positive and negative values of ψ roughly coincide with those of ω.

On the other hand, the nonlinear term $T[\nabla\psi]$ itself shows a markedly different pattern from that of ω or ψ; no correlation is noticeable with them. However, the potential term f, proportional to the difference $T[\nabla\psi] - |\nabla\psi|^2$, does show a strong correlation with ω. Namely, regions with large *negative* values of the potential f coincide with peripheral filament-like regions around the vortices, rather than centers of vortices themselves.

Because filaments are generated through the stretching effects by the nonlinear term, this shows that regions with strongly negative values of f collapse with near-singular structures with large vorticity gradient. In principle, near-singularities can be triggered by either strongly positive values of f as well as strongly negative values. It has turned out that it is the negative values which are responsible for generating near-singularities. It is of interest to note that in (10) the viscous term has a negative sign. This is a bit unexpected, as the viscous effect contributes to make f more negative.

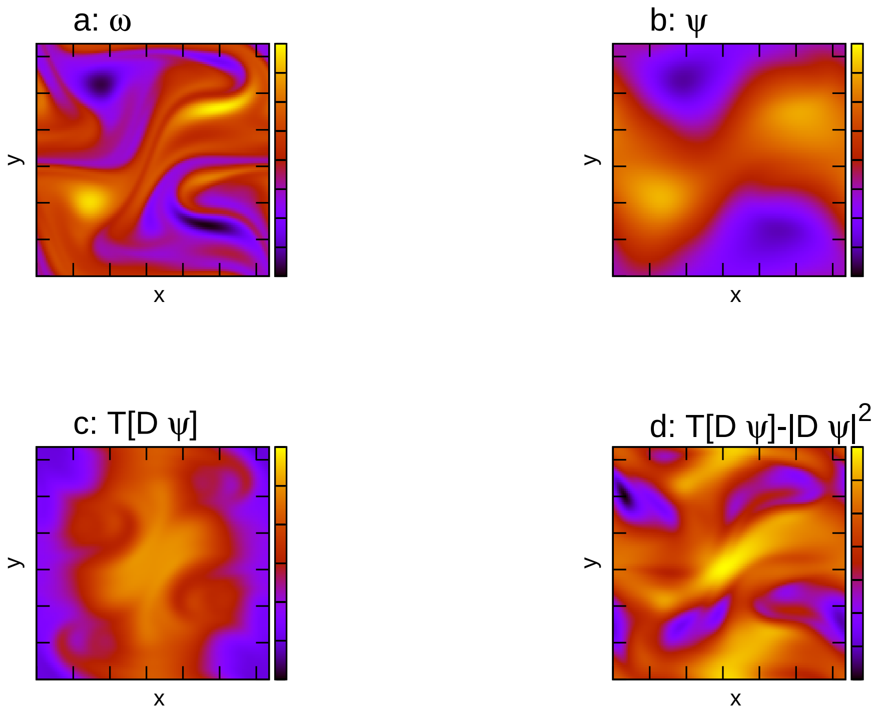

Figure 2. Flow fields at $t = 8$, a: ω, b: ψ, c: $T[\nabla\psi]$ and d: $T[\nabla\psi] - |\nabla\psi|^2$ for $\nu = 2 \times 10^{-3}$. Hereafter, positive values are shown in red or yellow and negative ones in blue.

In Fig.3 we show gradation plots at a late time $t = 20$ for the case of $N = 2048$, with the smaller viscosity $\nu = 1 \times 10^{-3}$. We observe two negative vortices about to merge in the center, whereas the pattern of ψ is smoothly located therein. The nonlinear term $T[\nabla\psi]$ shows no obvious correlation with any of those; it is positive somewhere and negative elsewhere in the center. As for the potential term f, it shows again a strong correlation with peripheral filaments surrounding the central vortices.

We can also confirm in Fig.4 the same features of the potential term f as above for the case of $N = 4096$ with the even smaller viscosity $\nu = 5 \times 10^{-4}$. The excess $T[\nabla\psi] - |\nabla\psi|^2$ captures near-singularities with steep vorticity gradient $|\nabla\omega|$. The probabilistic approach via the Feynman-Kac formula gives an alternative characterisation to the conventional interpretation by the effect of strong shear.

5. Cole-Hopf transforms for three-dimensional Navier-Stokes equations

In this section, we introduce an analogue of the Cole-Hopf transform for three-dimensional incompressible Navier-Stokes equations using a straightforward componentwise extension.[1]

Using the vector potentials $\boldsymbol{A}$ such that $\boldsymbol{u} = \nabla \times \boldsymbol{A}$ with the condition $\nabla \cdot \boldsymbol{A} = 0$, the three-dimensional Navier-Stokes equations written in $\boldsymbol{A}$ have been derived in

[1]Another approach is to write the vorticity and "tensor-potential" as skew-symmetric matrices and consider matrix logarithms. It appears that this does not work, because of non-commutative nature of matrix multiplications.

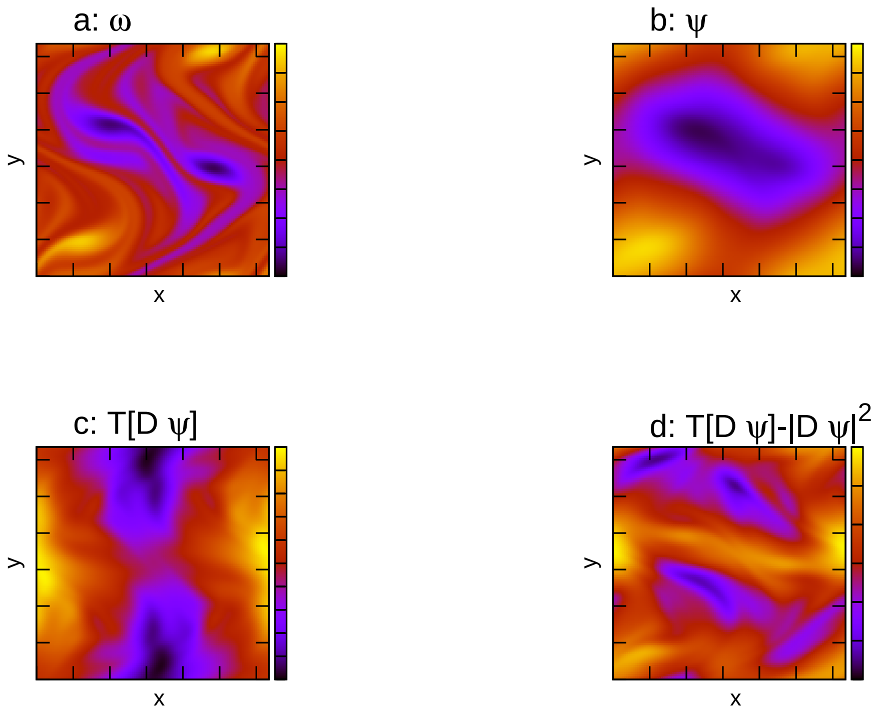

Figure 3. Flow fields at $t = 20$, a:ω, b:ψ, c: $T[\nabla\psi]$ and d:$T[\nabla\psi] - |\nabla\psi|^2$ for $\nu = 1 \times 10^{-3}$.

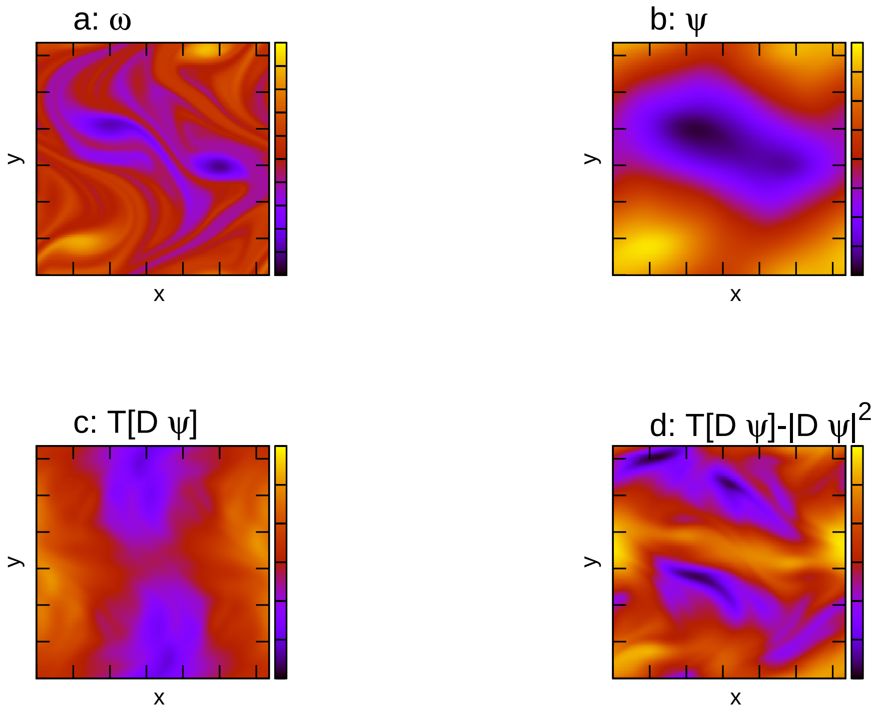

Figure 4. Flow fields at $t = 20$, a: ω, b: ψ, c: $T[\nabla\psi]$ and d:$T[\nabla\psi] - |\nabla\psi|^2$ for $\nu = 5 \times 10^{-4}$.

[10]

$$\frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{T}[\nabla \boldsymbol{A}] + \nu \triangle \boldsymbol{A}, \tag{13}$$

where

$$\boldsymbol{T}[\nabla \boldsymbol{A}] = \frac{3}{4\pi}\text{P.V.}\int_{\mathbb{R}^3} \frac{\boldsymbol{r} \times (\nabla \times \boldsymbol{A}(\boldsymbol{y}))\, \boldsymbol{r} \cdot (\nabla \times \boldsymbol{A}(\boldsymbol{y}))}{|\boldsymbol{r}|^5}\, \mathrm{d}\boldsymbol{y}, \tag{14}$$

with $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$. We note that $\nabla \cdot \boldsymbol{T}[\nabla \boldsymbol{A}] = 0$ holds.

We introduce the Cole-Hopf transforms componentwise for each $\theta_j > 0$,

$$A_j = k \log \theta_j, \quad (j = 1, 2, 3), \tag{15}$$

where

$$\frac{1}{\theta_1}\frac{\partial \theta_1}{\partial x_1} + \frac{1}{\theta_2}\frac{\partial \theta_2}{\partial x_2} + \frac{1}{\theta_3}\frac{\partial \theta_3}{\partial x_3} = 0.$$

By substituting (15) into (13), we find

$$\frac{\dfrac{\partial \theta_j}{\partial t} - \nu \triangle \theta_j}{\theta_j} = \underbrace{kT_j\left[\frac{\nabla \theta_1}{\theta_1}, \frac{\nabla \theta_2}{\theta_2}, \frac{\nabla \theta_3}{\theta_3}\right] - \nu \frac{|\nabla \theta_j|^2}{\theta_j^2}}_{\equiv f_j(\boldsymbol{x}, t)}.$$

Hence we obtain a system of heat equations with the potential term

$$\frac{\partial \theta_j}{\partial t} = \nu \triangle \theta_j + f_j(\boldsymbol{x}, t)\theta_j, \quad \text{(no summation)} \tag{16}$$

where

$$f_j(\boldsymbol{x}, t) = kT_j\left[\frac{\nabla \theta_1}{\theta_1}, \frac{\nabla \theta_2}{\theta_2}, \frac{\nabla \theta_3}{\theta_3}\right] - \nu \frac{|\nabla \theta_j|^2}{\theta_j^2}, \quad (j = 1, 2, 3).$$

It is clear that if f_j's are bounded $\int_0^t \sup_{\boldsymbol{x},j} |f_j(\boldsymbol{x}, s)| ds < \infty$, that is,

$$\int_0^t \sup_{\boldsymbol{x}} |\boldsymbol{f}(\boldsymbol{x}, s)| ds < \infty,$$

where $|\boldsymbol{f}| = \sqrt{f_1^2 + f_2^2 + f_3^2}$, the solutions θ_j of (16) are smooth on the time interval. We thus obtain

$$\theta_j(\boldsymbol{x}, t) = E\left[\theta_j(\boldsymbol{B}_t, 0)\exp\left(\int_0^t f_j(\boldsymbol{B}_s, s) ds\right)\right], \quad \text{(no summation)}$$

for each $j = 1, 2, 3$ by applying the Feynman-Kac formula. This generalises (11). We refer the reader to **Appendix** C for a further extension to n-dimensional cases.

6. Summary and discussion

We have introduced a generalisation of the Cole-Hopf transform by recasting the Navier-Stokes equation as a path integral form using the Feynman-Kac formula. For the two-dimensional case, this reads for $\theta = \exp(\psi/k)$,

$$\theta(\boldsymbol{x}, t) = E\left[\theta_0(\boldsymbol{B}_t)\exp\left(\int_0^t f(\boldsymbol{B}_s, s)ds\right)\right],$$

where

$$f(\boldsymbol{x}, t) = k\,T\left[\frac{\nabla\theta}{\theta}\right] - \nu\frac{|\nabla\theta|^2}{\theta^2}.$$

It is reminiscent of the solution of the forced Burgers equation via the Cole-Hopf transform.

Using numerical experiments, we have shown how the potential term f successfully captures the near-singularities in turbulent flow fields. In two-dimensional Navier-Stokes flows they take the form of filaments with large vorticity gradient, where intense stretching of vorticity contours taking place under strong shear. This probabilistic approach offers an alternative view or an advantage, where the residual of cancellation of nonlinear terms gives rise to a regularity condition. In particular, we have found that it is the *negative* values of the potential term that are responsible for near-singularity associated with large vorticity gradient.

Let us consider the benefits of this approach. For the three-dimensional case, we recall the regularity criteria for the Navier-Stokes equations:

$$\int_0^t \|\boldsymbol{u}\|_{L^\infty}^2 ds < \infty \text{ or } \int_0^t \|\boldsymbol{T}[\nabla\boldsymbol{A}]\|_{L^\infty} ds < \infty \Longrightarrow \text{smoothness up to time } t, \tag{17}$$

where the former one is classical [11], while the latter is due to [12]. In the current approach, the following also holds true:

$$\int_0^t \|\boldsymbol{f}\|_{L^\infty} ds < \infty, \text{ for some } k(\neq 0) \Longrightarrow \text{smoothness up to } t, \tag{18}$$

because once the condition (18) is satisfied for *one* value of k, the Feynman-Kac theorem assures that the solution is smooth on the time interval. Hence, the condition $\int_0^t \|\boldsymbol{f}\|_{L^\infty} ds < \infty$ also serves as a regularity criterion. By the definition

$$f_j = \frac{1}{k^2}\left(kT_j[\nabla\boldsymbol{A}] - \nu|\nabla A_j|^2\right),$$

and $\sum_{i,j=1}^3 (\partial_i A_j)^2 = k^2\sum_{j=1}^3 |\nabla\theta_j|/\theta_j^2 = |\boldsymbol{u}|^2/2$, we see (18) contains new information. Actually, as a contraposition to (18) we have equivalently

$$\text{blowup at time } t \Longrightarrow \int_0^t \|\boldsymbol{f}\|_{L^\infty} ds = \infty, \text{ for } all\ k(\neq 0). \tag{19}$$

Because $\sup_{\boldsymbol{x}} |\boldsymbol{f}|$ and $\sup_{\boldsymbol{x},j} |f_j|$ are comparable, if a singularity exists, it is impossible for f to stay bounded no matter how carefully we choose k. The unbounded integrals in the above two criteria (17) resist sufficiently substantial cancellations that can make their summation bounded, which is an additional characterisation of possible singularities. A possible interpretation of this result can be the effect of phase-shift of the singular integral $\boldsymbol{T}[\nabla \boldsymbol{A}]$: when $|\nabla \boldsymbol{A}|$ has an unimodal local maximum, $\boldsymbol{T}[\nabla \boldsymbol{A}]$ tends to have a kink-like structure with an offset of its local maxima, making their local maxima dislocated.

It may be in order to compare the current result with that of [13]. The Cauchy invariant is a well-known first integral for the Euler equations in vorticity form. The vortex lines are material in inviscid fluids, but no longer so in viscous fluids. In [13], its dissipative counterpart for the Navier-Stokes equations has been presented and local existence proved. This was done by a sophisticated analysis which makes nontrivial use of the nonlinear term. It is intriguing to observe that a similar formula holds for the vorticity in viscous fluids (See also [14].)

Here, the generalisation of the Cole-Hopf transform, followed by the application of the Feynman-Kac formula is straightforward and it is valid for any quadratic nonlinear term when k has the same dimension as the kinematic viscosity ν. A benefit of this approach is that a regularity criterion similar to Serrin's, but with a new element, immediately follows from the boundedness condition on the potential term without effort.

To clarify this point it is instructive to recall the standard integral equations for the Navier-Stokes equations. Using the heat kernel, we can apply the Duhamel principle to the Navier-Stokes equations to get

$$\boldsymbol{u}(t) = G_t * \boldsymbol{u}_0 - \int_0^t G_{t-s} * \mathbb{P}\, \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) ds,$$

where $\mathbb{P} = \boldsymbol{I} - \nabla \triangle^{-1} \nabla \cdot$ denotes a solenoidal projection. It is well-known that a successive approximation based on the above integral equations is known to converge on a time interval where the velocity remains bounded. However, we cannot deduce such a result immediately by taking a glimpse at the integral equations; it requires some nontrivial efforts to deduce it.

An application of this approach to the numerical study of three-dimensional turbulence is of interest, just as we have done in two dimensions. It is also of interest to seek further study on the possible singularities on the basis of the Cole-Hopf transforms [12, 15]. These will be left for future study.

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Appendix A. The pressure and the stream function in two dimensions

The following expression of the nonlinear term

$$\begin{aligned} r &\equiv T[\nabla \psi] \\ &= \epsilon_{jk} R_i R_j \partial_k \psi \partial_i \psi \\ &= (R_1 R_1 - R_2 R_2) \partial_1 \psi \partial_2 \psi - R_1 R_2 [(\partial_1 \psi)^2 - (\partial_2 \psi)^2], \end{aligned} \tag{A1}$$

is useful in numerical experiments, where $R_j \equiv -\partial_j (-\triangle)^{-1/2}$ denotes the Riesz transforms and $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$. It should be noted that it is similar to, but different from the pressure, which is given by

$$\begin{aligned} p &= (-\triangle)^{-1}\frac{\partial^2 u_i u_j}{\partial x_i \partial x_j} \\ &= R_i R_j u_i u_j = R_i R_j \epsilon_{ik}\partial_k\psi\,\epsilon_{jl}\partial_l\psi \\ &= R_1R_1(\partial_2\psi)^2 + R_2R_2(\partial_1\psi)^2 - 2R_1R_2\partial_1\psi\partial_2\psi. \end{aligned}$$

Noting that p has two skewed gradients and r has one, it makes sense to introduce yet another quadratic form with *no* skewed gradients

$$q = R_iR_j\partial_i\psi\partial_j\psi = R_1R_1(\partial_1\psi)^2 + R_2R_2(\partial_2\psi)^2 + 2R_1R_2\partial_1\psi\,\partial_2\psi.$$

Clearly we have

$$p + q = -(\partial_1\psi)^2 - (\partial_2\psi)^2 = -|\boldsymbol{u}|^2 \leq 0.$$

Transformations between (r, p, q) and $((\partial_1\psi)^2, (\partial_2\psi)^2, \partial_1\psi\,\partial_2\psi)$ is given by

$$\begin{pmatrix} r \\ p \\ q \end{pmatrix} = \underbrace{\begin{pmatrix} -R_1R_2 & R_1R_2 & R_1^2 - R_2^2 \\ R_2^2 & R_1^2 & -2R_1R_2 \\ R_1^2 & R_2^2 & 2R_1R_2 \end{pmatrix}}_{\equiv A} \begin{pmatrix} (\partial_1\psi)^2 \\ (\partial_2\psi)^2 \\ \partial_1\psi\,\partial_2\psi \end{pmatrix}.$$

Because $\det A = -(R_1^2 + R_2^2)^3 = 1 \neq 0$ and it is just a number rather than an operator, we can invert the above relationship to obtain

$$\begin{pmatrix} (\partial_1\psi)^2 \\ (\partial_2\psi)^2 \\ \partial_1\psi\,\partial_2\psi \end{pmatrix} = \begin{pmatrix} -2R_1R_2 & R_2^2 & R_1^2 \\ 2R_1R_2 & R_1^2 & R_2^2 \\ R_1^2 - R_2^2 & -R_1R_2 & R_1R_2 \end{pmatrix} \begin{pmatrix} r \\ p \\ q \end{pmatrix},$$

as verified by direct computations. In terms of p, q and r, the potential term can be written

$$f(\boldsymbol{x}, t) = \frac{1}{k^2}\left(kr + \nu(p+q)\right).$$

We observe that f is similar to p, but they are not directly proportional to each other.

Appendix B. Numerical details

We solve the two-dimensional Navier-Stokes equation (6) using a standard Fourier pseudo-spectral method. In order to estimate $T[\nabla\psi]$, we note by (A1) that (7) can be written equivalently as follows [5]

$$\frac{\partial\psi}{\partial t} = \epsilon_{jk}R_iR_j\partial_k\psi\partial_i\psi + \nu\triangle\psi.$$

The Fourier transform of the nonlinear term $\widehat{T[\nabla\psi]}$ can be written

$$\widehat{T[\nabla\psi]} = -\frac{k_1^2 - k_2^2}{|\boldsymbol{k}|^2}\widehat{\partial_1\psi\partial_2\psi} + \frac{k_1 k_2}{|\boldsymbol{k}|^2}\left(\widehat{(\partial_1\psi)^2} - \widehat{(\partial_2\psi)^2}\right),$$

where $\boldsymbol{k} = (k_1, k_2) \neq 0$ is the wavenumber. The right-hand side of the above expression can be evaluated easily by estimating the convolution products.

Appendix C. Cole-Hopf transforms for n-dimensional Navier-Stokes equations

We extend the Cole-Hopf transform to the Navier-Stokes equations in $\mathbb{R}^n$ $(n \geq 2)$

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \nu \triangle u_i. \ (i = 1, 2, \ldots, n) \tag{C1}$$

In this case, we consider the vorticity tensor $\omega_{ij} = \dfrac{\partial u_j}{\partial x_i} - \dfrac{\partial u_i}{\partial x_j}$ and the tensor potential ψ_{ij} $(i, j = 1, 2, \ldots, n)$, which are related by $\omega_{ij} = -\triangle\psi_{ij}$ where $\triangle = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$. The governing equations for ψ_{ij} read [15]

$$\frac{\partial \boldsymbol{\psi}}{\partial t} = \boldsymbol{T}[\nabla\boldsymbol{\psi}] + \nu\triangle\boldsymbol{\psi},$$

where

$$\begin{aligned} T_{ij}[\nabla\boldsymbol{\psi}](\boldsymbol{x}) \equiv & -\frac{1}{\sigma_n}\text{P.V.}\int_{\mathbb{R}^n}\left(\frac{\delta_{ki}}{r^n} - \frac{r_k\, r_i}{r^{n+2}}\right)\frac{\partial\psi_{kl}}{\partial y_l}\frac{\partial\psi_{jm}}{\partial y_m}d\boldsymbol{y} \\ & + \frac{1}{\sigma_n}\text{P.V.}\int_{\mathbb{R}^n}\left(\frac{\delta_{kj}}{r^n} - \frac{r_k\, r_j}{r^{n+2}}\right)\frac{\partial\psi_{kl}}{\partial y_l}\frac{\partial\psi_{im}}{\partial y_m}d\boldsymbol{y}, \end{aligned}$$

with $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$, $\sigma_n = \dfrac{2\pi^{n/2}}{\Gamma(n/2)}$ and Γ is the gamma function. By introducing the Cole-Hopf transform componentwise

$$\psi_{ij} = k\log\theta_{ij}, \ \ (i, j = 1, 2, \ldots, n)$$

we find

$$\frac{\dfrac{\partial\theta_{ij}}{\partial t} - \nu\triangle\theta_{ij}}{\theta_{ij}} = \underbrace{kT_{ij}\left[\frac{\nabla\theta_{12}}{\theta_{12}}, \ldots, \frac{\nabla\theta_{n-1\,n}}{\theta_{n-1\,n}}\right] - \nu\frac{|\nabla\theta_{ij}|^2}{\theta_{ij}^2}}_{\equiv f_{ij}(\boldsymbol{x},t)}.$$

We thus obtain

$$\frac{\partial\theta_{ij}}{\partial t} = \nu\triangle\theta_{ij} + f_{ij}(\boldsymbol{x}, t)\theta_{ij}, \ \ \text{(no summation)}$$

where

$$f_{ij}(\boldsymbol{x}, t) = kT_{ij}\left[\frac{\nabla\theta_{12}}{\theta_{12}}, \ldots, \frac{\nabla\theta_{n-1\,n}}{\theta_{n-1\,n}}\right] - \nu\frac{|\nabla\theta_{ij}|^2}{\theta_{ij}^2}. \ (i, j = 1, 2, \ldots, n).$$

As before, the boundedness condition

$$\int_0^t \sup_{\boldsymbol{x},i,j} |f_{ij}(\boldsymbol{x},s)| ds < \infty$$

can be identified as a regularity criterion. Under this condition, the Feynman-Kac formula yields

$$\theta_{ij}(\boldsymbol{x},t) = E\left[\theta_{ij}(\boldsymbol{B}_t,0)\exp\left(\int_0^t f_{ij}(\boldsymbol{B}_s,s)ds\right)\right], \quad (i,j=1,2,\ldots,n) \text{ (no summation)},$$

where $\boldsymbol{B}_t$ denotes an n-dimensional Brownian motion.

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